○ instanton

$X = \mathbb{R}^4 = \mathbb{C}^2$

x_0, x_1, x_2, x_3 $\qquad z = x_0 + i x_1, \; w = x_2 + i x_3$

E : C^∞ vector b'dle on $\mathbb{R}^4$ with a Hermitian metric

$= \mathbb{R}^4 \times \mathbb{C}^r$

A : unitary connection

$= A_0\, dx_0 + A_1\, dx_1 + A_2\, dx_2 + A_3\, dx_3$

A_α : $u(r)$-valued function on X

↳ skew Hermitian matrices

$d_A = d + A \wedge \cdot \quad : C^\infty(\Lambda^p T^*X \otimes \mathbb{C}^r) \to C^\infty(\Lambda^{p+1} T^*X \otimes \mathbb{C}^r)$

$d_A^2 = \underbrace{F_A}_{\shortparallel} \wedge \cdot$: curvature, $u(r)$-valued 2-form

$dA + A \wedge A$

• gauge transformation

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr] & & E \arrow[dl] \\ & X & \end{tikzcd} \qquad g : X \to U(r)$$

$$\boxed{A^g = g^{-1} dg + g^{-1} A g}$$

$$d_{A^g}\varphi = g^{-1} d_A(g\varphi) = g^{-1}\left[d(g\varphi) + A(g\varphi)\right]$$

$$= d\varphi + \underbrace{(g^{-1}dg + g^{-1}Ag)}_{\shortparallel \; A^g}\varphi$$

Now we regard $X = \mathbb{C}^2 \ni z, w$

$d\bar{z} = dx_0 - i dx_1$, $d\bar{w} = dx_2 - i dx_3$ etc

$$A = \frac{1}{2}\Big[\underbrace{(A_0 + iA_1)\,d\bar{z} + (A_2 + iA_3)\,d\bar{w}}_{A^{0,1}} + \underbrace{(A_0 - iA_1)\,dz + (A_2 - iA_3)\,dw}_{A^{1,0}}\Big]$$

A defines a holomorphic structure (satisfies integrability condition)

$\overset{\text{def.}}{\Longleftrightarrow}$ $\bar{\partial}_A^{\,2} = 0$ $\qquad d_A = \partial_A + \bar{\partial}_A = \partial + A^{1,0} + \bar{\partial} + A^{0,1}$

(0,2)-part of the curvature F_A

$\partial_A^2 = 0$ is satisfied automatically

$$\begin{aligned}
dz \wedge dw &= (dx_0 + i dx_1) \wedge (dx_2 + i dx_3) \\
&= dx_0 \wedge dx_2 - dx_1 \wedge dx_3 + i(dx_1 \wedge dx_2 + dx_0 \wedge dx_3)
\end{aligned}$$

$$\begin{cases}
dz \wedge d\bar{z} = 2i\, dx_0 \wedge dx_1 \\
dw \wedge d\bar{w} = 2i\, dx_2 \wedge dx_3 \\
dz \wedge d\bar{w} = (dx_0 + i dx_1) \wedge (dx_2 - i dx_3) \\
\qquad = dx_0 \wedge dx_2 + dx_1 \wedge dx_3 - i(dx_0 \wedge dx_3 - dx_1 \wedge dx_2) \\
d\bar{z} \wedge dw = dx_0 \wedge dx_2 + dx_1 \wedge dx_3 + i(dx_0 \wedge dx_3 - dx_1 \wedge dx_2)
\end{cases}$$

<u>Lemma</u> A : holomorphic

$\Longleftrightarrow$ $F_A = \sum_{\alpha<\beta} F_{\alpha\beta}\, dx_\alpha \wedge dx_\beta$ satisfies

$$F_{02} = F_{13} \quad , \quad F_{03} = -F_{12}$$

Now consider $X = \mathbb{H}$ quaternion $x_0 + ix_1 + jx_2 + kx_3$

Then $\quad i^2 = j^2 = k^2 = -1$

$$X \underset{\uparrow}{\cong} \mathbb{C}^2$$

3 choices $i, j, k \leftrightarrow \sqrt{-1}$ $\quad ijk = -1$

$$\begin{aligned} x_0 + ix_1 + jx_2 + kx_3 &= (x_0 + ix_1) + (x_2 + ix_3)j \\ &= (x_0 + jx_2) + (x_3 + jx_1)k \\ &= (x_0 + kx_3) + (x_1 + kx_2)i \end{aligned}$$

Def. A: anti-self-dual (or instanton)

$\iff \bar{\partial}_A^2 = 0$ for any cpx str. i, j, k on X

$$\begin{aligned} d(x_0 + jx_2) \wedge d(x_3 + jx_1) = {}& dx_0 \wedge dx_3 - dx_2 \wedge dx_1 \\ & + j(dx_0 \wedge dx_1 + dx_2 \wedge dx_3) \end{aligned}$$

$$\begin{aligned} d(x_0 + kx_3) \wedge d(x_1 + kx_2) = {}& dx_0 \wedge dx_1 - dx_3 \wedge dx_2 \\ & + k(dx_0 \wedge dx_2 + dx_3 \wedge dx_1) \end{aligned}$$

Lemma. ASD $\iff F_A$ satisfies

$$\begin{cases} F_{01} = -F_{23} \\ F_{02} = -F_{31} \\ F_{03} = -F_{12} \end{cases} \qquad (F_{\alpha\beta} = -F_{\beta\alpha})$$

Hodge star operator: $\quad * : \Lambda^2 T^* X \circlearrowleft \qquad *^2 = 1$

$* dx_0 \wedge dx_1 = dx_2 \wedge dx_3 \quad$ etc

Lemma.

instanton $\iff *F_A = -F_A$ anti-self-duality equation

Motivation

Study moduli spaces

$= \left\{ \text{gauge equiv. classes of sol's of } \begin{matrix} \text{instantons} \\ \text{monopoles} \end{matrix} \right\}$

— We need to specify bdry condition

... will be done later

○ monopole

$= \mathbb{R}$-invariant instanton

$$\mathbb{R}^4 \ni \underbrace{x_0}_{t}, \underbrace{x_1, x_2, x_3}_{=: x}$$

$$A = A_0\, dx_0 + \cdots + A_3\, dx_3$$
$$= \Phi(x)\, dt + \underbrace{A_1(x)\, dx_1 + \cdots + A_3(x)\, dx_3}_{A' : \text{ connection on } \mathbb{R}^3}$$

$\therefore$ A in 4D inv. $\longleftrightarrow$ A' in 3D + $\Phi \in \mathfrak{u}(r)$-valued function

$$F_A = dA + A \wedge A$$
$$= F_{A'} + \underbrace{\sum_\alpha \frac{\partial \Phi}{\partial x_\alpha} dx_\alpha \wedge dt + (A_\alpha \wedge \Phi - \Phi \wedge A_\alpha) dx_\alpha \wedge dt}_{= d_{A'}\Phi \wedge dt}$$

Lemma.

$$*_4 F_A = -F_A \iff F_{A'} = *_3\, d_{A'}\Phi$$

Bogomolny equation

∵ coef. of $dx_1 \wedge dt = \dfrac{\partial \Phi}{\partial x_1} + [A_1, \Phi]$

$\qquad ||$

coef. of $dx_2 \wedge dx_3 = (F_{A'})_{23}$

$$*_3\, dx_1 = dx_2 \wedge dx_3$$

//

Def. A : connection on $\mathbb{R}^3$
Φ : section of $P \times_{Ad} \mathfrak{u}(r)$

(A, Φ) is a monopole $\Longleftrightarrow$ $F_A = * d_A \Phi$

$r = 1 \Rightarrow \Phi$: $i\mathbb{R}$-valued fct. & $d_A \Phi = d\Phi$

$$0 = \underbrace{dF_A}_{\text{Bianchi identity}} \quad (F_A = dA) \quad = d * d\Phi = * \Delta \Phi$$

$\therefore \Phi$: harmonic function

maximal principle : Φ : bdd $\Longrightarrow$ $\Phi = \text{const}$
$F = 0$
(trivial connection)

Th. $U(1)$-monopole with no singularity $\Longrightarrow$ trivial
(& bdd Φ)

<u>Dirac monopole</u>

$$i\Phi = 1 + \frac{k}{2r} \qquad r = |x|, \quad k = \text{const} \qquad \Rightarrow \Delta \Phi = 0 \text{ except } x = 0$$

$$F_A = * d\Phi = -\frac{ik}{2r^2} * dr$$

Chern-Weil theory $\Rightarrow$ $\frac{i}{2\pi} F_A$ represents $c_1(E)$

$$\therefore \mathbb{Z} \ni \int_{S^2} \frac{i}{2\pi} F_A = \frac{k}{4\pi} \underbrace{\int_{S^2} * dr}_{= 4\pi} = k$$

<u>Exercise</u> If $k \in \mathbb{Z} \Rightarrow E$ & A exists (outside $x = 0$)

○ Nahm's equation

$\mathbb{R}^3$-inv. instanton

$$A = A_0(t)dt + T_1(t)dx_1 + T_2(t)dx_2 + T_3(t)dx_3$$

$$F_A = \sum \frac{dT_\alpha}{dt} dt \wedge dx_\alpha + [A_0, T_\alpha]\, dt \wedge dx_\alpha$$

$$+ [T_\beta, T_\gamma]\, dx_\beta \wedge dx_\gamma$$

$$\therefore \quad \frac{\nabla}{dt} T_1 + [T_2, T_3] = 0 \quad (1 \to 2 \to 3)$$

Nahm's equation

Rem. T_0 can be absorbed to ∇ by a gauge transformation

$$\text{Solve } g : A_0^g = g^{-1}\frac{d}{dt} g + g^{-1} A_0 g = 0$$

$$\therefore \quad \frac{d}{dt} T_1 + [T_2, T_3] = 0 \qquad \text{etc}$$

○ Hitchin's self-duality equation

$\mathbb{R}^2$-inv.

$$A = \underbrace{A_0\, dx_0 + A_1 dx_1}_{= A'} + \Phi_2\, dx_2 + \Phi_3\, dx_3 \qquad z = x_0 + i x_1$$

$$F_A = F_{A'} + \frac{\partial \Phi_2}{\partial x_0} dx_0 \wedge dx_2 + \frac{\partial \Phi_2}{\partial x_1} dx_1 \wedge dx_2 + \frac{\partial \Phi_3}{\partial x_0} dx_0 \wedge dx_3$$

$$+ \frac{\partial \Phi_3}{\partial x_1} dx_1 \wedge dx_3 + [A_0, \Phi_2] dx_0 \wedge dx_2 + [A_0, \Phi_3] dx_0 \wedge dx_3$$

$$+ [A_1, \Phi_2]\, dx_1 \wedge dx_2 + [A_1, \Phi_3]\, dx_1 \wedge dx_3$$

$$+ [\Phi_2, \Phi_3]\, dx_2 \wedge dx_3$$

Set $\Phi = (\Phi_2 + i\Phi_3)dz$ $\qquad z = x_0 + ix_1$

$$\begin{cases} F_{A'} = \frac{1}{2}[\Phi, \Phi^*] \\ \nabla_{\frac{\partial}{\partial \bar{z}}} \Phi = 0 \end{cases}$$

$F_{A'}$'s (1.2)-part $+ [\Phi_2, \Phi_3] = 0$

$\Phi^* = (-\Phi_2 - i\Phi_3)d\bar{z}$

$[\Phi, \Phi^*] = [\Phi_2, \Phi_3]\, dz \wedge (-i)\, d\bar{z} = -i\,[\Phi_2, \Phi_3]\, dz \wedge d\bar{z}$

$$dz \wedge d\bar{z} = (dx_0 + i\,dx_1) \wedge (dx_0 - i\,dx_1)$$
$$= -2i\, dx_0 \wedge dx_1$$

$\nabla_{\frac{\partial}{\partial x_0}} \Phi_2 = \nabla_{\frac{\partial}{\partial x_1}} \Phi_3$ $\qquad \nabla_{\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1}} \Phi_2 = \nabla_{\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_0}} \Phi_3$

$\nabla_{\frac{\partial}{\partial x_1}} \Phi_2 = -\nabla_{\frac{\partial}{\partial x_0}} \Phi_3$ $\qquad = -i\, \nabla_{\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1}} \Phi_3$

$\nabla_{\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1}} (\Phi_2 + i\Phi_3)$